BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE

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VOL. XXV

OCTOBER 1918 TO JULY 1919

Published by the
AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island 02904
1919

Reprinted 1969

Whole No. 271

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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

A CONSPECTUS OF THE MODERN THEORY OF DIVERGENT SERIES.*

BY PROFESSOR WALTER B. FORD.

IT will doubtless be recalled by all here present that the first volume of the encyclopedia of mathematics, as originally published in German, was issued in hefts, or parts, each prepared by a special collaborator well qualified for his task. The completion of the volume occupied a period of about six years and resulted in a finished work of no less than 1,128 pages. This was followed some years later by the second volume, which was still larger, containing in all its parts 1,154 pages. The two volumes taken together were intended to cover the entire range of pure mathematical analysis (as distinguished from geometry, mechanics, and other applications), each important branch being treated in its essentials and the amount of space allotted to any one topic being proportionate, at least roughly, to its relative importance. Out of the grand total of 2,282 pages thus presented it may now be remarked that a little less than 7 pages were devoted to that particular topic specified as "divergent series." Thus we have a ratio of 7 to 2,282, or about three tenths of one per cent, which may fairly be taken as the measure of interest in this topic at the time when the encyclopedia began to appear; that is, in the neighborhood of twenty years ago. The 7 pages in question, as we come to examine them, seem directed mainly to showing by means of simple illustrations that the processes by which Euler arrived at certain noteworthy results while

* Address of the retiring chairman of the Chicago Section of the American Mathematical Society, read at the joint meeting of the Section and of the Mathematical Association of America at Chicago, December 28, 1917.

dealing with divergent series are in themselves altogether unjustified and unscientific, the correctness of his results being in the nature of a happy circumstance arising out of the inherent character of the particular series he happened to be dealing with. On the whole, the short article conveys a rather gloomy outlook for this entire field of study, especially as regards any attempts that may be made to put it upon a truly scientific basis. Without digressing further upon the attitude of the encyclopedia, I take for granted that to-day we are all quite willing to agree that, contrary to any predictions that may have been made in the past, divergent series have now come to occupy a prominent place in analysis and one that bids fair to be permanent. How, it may be asked, has this happened? What has been done during these twenty years that really constitutes a vital advance in this field of study? Can we say that divergent series are at last upon a scientific basis? These are fair questions and it is to them that I would respectfully direct your attention for a few moments this afternoon. In answering, I shall attempt no more than an outline or conspectus of the situation as I myself have come to regard it and in this I am conscious beforehand that my own feelings, at least at some points, may not be universally acceptable; yet I shall venture all with equal candor. I shall not attempt to detail a large body of more or less intricate theorems and results, but I shall endeavor in a general way to show what seems central to me both as to the logical position of the present day theory and its applications.

In the first place, if I may refer again to Euler, we should of course recognize clearly at the outset that in his day the class of series which we now call divergent had not been clearly separated off by itself and in this sense it really had no well-defined meaning. It is true that certain individuals of the class had been studied more or less extensively and in ways which we are now bound to regard as interesting because of their curiosity, as for example the assignment of the sum $\frac{1}{2}$ to the oscillating divergent series

$$1-1+1-1+1-1+\ \ldots,$$

but divergent series as a class had not been carefully defined. Not until the time of Cauchy and Abel did they take on an exact sense and it was then through a purely negative process.

In fact, they were then defined, as they still are to-day, as being all those series which do *not* satisfy the particular definition for convergent series which was formulated by these two great mathematicians. On this plan, convergence means nothing more or less than that the sum of the first n terms approaches a limit as n increases indefinitely, while divergence means simply that this particular limit does not exist. However natural this sort of a classification of series may seem to us, owing to our having been born and brought up with it, we must recognize that it involves after all a large element of arbitrariness. In fact, the only reason why the fundamental distinction between series should be made to hinge upon the existence of this particular limit lies in the fact that those series for which it exists are an admittedly important class. Of course there is no adequate reason in this to account for the way in which the excluded or divergent class were so long held in utter disrepute by the successors of Abel and Cauchy and continue still to repel us, except it be the psychological fact that whatever is excluded from a certain good class we instinctively think of as forming a bad class. However, to banish a whole class of series from analysis simply because it does not bear the stamp of a certain branded variety may easily be and is too hasty and rough a handling of their case. It makes no allowance for the different degrees of respectability which, though divergent, they may still possess. Thus it is, in substance, that the modern studies on divergent series may be said to have arisen. They are a reaction against the underlying arbitrariness of the Abel-Cauchy distinction and the ruthless and unjustified exclusions which this particular distinction has tended to produce. If we inquire just what the avenue of approach has been to this subject, it may be said that the first tangible result was the creation of the so-called "sum formulas," the essential feature of any such formula being that it shall not only serve, in case it can be evaluated, to give us the sum of any convergent series, but it shall continue to preserve a meaning when applied in the same way to certain divergent series, thus associating, or assigning, definite numerical values or sums to them also. The spirit of this procedure is, of course, nothing more or less than that common to all extensions of idea in mathematics. Just as in the theory of functions of a complex variable the sine of x, or any other function such as there considered, comes to have a

meaning for complex values of x only by virtue of a certain definitional formula so constructed as to give results when x is real that are in accord with the known properties of the sine in the real domain, so in the modern studies on divergent series sums are assigned to such series only by virtue of sum formulas so constructed as to yield results when applied to convergent series that are in accord with the known facts regarding such series. I have purposely mentioned this particular illustration because by carrying it one step farther, as I shall now do, I shall be able to bring out another feature of the present day situation as regards divergent series, and this time we shall discover that the theory is by no means in a perfectly satisfactory logical state as yet, being in this respect quite different from any well established body of doctrine such as the theory of functions of a complex variable. In fact, it will be recalled that it is a vital feature of the complex variable theory that no two different definitions for any one given function, as sin x, are possible. In other words, it is demonstrable that if any two definitions for the function agree throughout the real domain, they will necessarily agree also throughout the complex domain. This property of uniqueness, if we may so designate it, is brought about by imposing a fundamental limitation at the very outset of the theory, namely, that only those functions shall be retained which belong to the class known as monogenic, or analytic. Thus, the logical coherence of the complex variable theory is bought at the price of a far reaching initial restriction, yet one which, as we know, is not so serious but what we are left with a class of functions of great interest in themselves and in their applications. In the theory of divergent series, on the other hand, as it exists today we are in possession of a large number of sum formulas each applicable in the sense before described to a divergent series and all agreeing with each other so far as convergent series are concerned, but *not* agreeing in general in the sums they prescribe to a given divergent series. In a word, the situation is analogous to that which the theory of functions would present in case the fundamental limitation as to the character of the functions considered were to be removed. How, you will at once inquire, can there be any general theory worthy of the name on such a loose plan as this? Strictly speaking, there cannot. However, all that is lacking in order that we have a bona fide theory is that we come to some agree-

ment, as in the theory of functions of a complex variable, as to what is proper by way of initial limitations, either upon the kind of divergent series to be considered or upon the kind of sum formulas we shall employ in connection with them. Have we any good indications as to what such limitations may well be? Yes, but none that are universally accepted as yet. As to what the line of approach to this matter may properly be I shall have more to say eventually, but for the moment I deem it more proper that I attempt to answer another question which no doubt by this time has arisen in your own minds. How can it be that divergent series have come into prominence in the face of such conditions as I have just described, whereby there does not exist even to this day a strictly coherent and universally acceptable general theory of the subject? This is a very proper question, whose answer is twofold.

First, the various sum formulas, regardless of their interrelations and other such logical aspects, have been found to yield interesting information when applied to certain important special series, such as Fourier's series, Dirichlet's series, etc. For example, even though the Fourier series representing a given function $f(x)$ may be divergent at a certain point, still the series may be summable there by one or more formulas and the sum thus obtained may and usually does continue to serve fully as useful a purpose from the standpoint of mathematical physics or other applications as does the sum when the same series is convergent; that is, the sum in the extended sense furnishes the answer to the proposed physical problem. Extensive investigations have been carried out in this connection to determine sufficient conditions under which a Fourier series will be summable, analogous to the well-known conditions for its convergence, and similarly the problem has been carefully worked over for some of the other related developments such as those for an arbitrary function in terms of Bessel functions, or Legendre functions. In this connection it may be of interest to observe the central fact that whereas the Fourier series for $f(x)$ converges in general to the value

$$\frac{f(x-0)+f(x+0)}{2}$$

only in case $f(x)$ is of limited total fluctuation in the neighborhood of the point x under consideration, the same series will

be summable by the simplest of the well known sum formulas of Cesàro to this same value provided only that the right and left limits, namely $f(x+0)$ and $f(x-0)$, exist. Here we incidentally meet with an illustration of the manner in which summability includes convergence as a special case, since the latter set of conditions is evidently much less restrictive than those just mentioned for convergence. And it may be added that, so far as the Bessel expansions are concerned, much the same situation prevails as with Fourier series, but the case of the Legendre expansions is essentially different and all the more interesting because of its novelty. Here no new results follow so long as one uses the simplest of the Cesàro formulas; that is, so long as one uses the formula of order 1 these developments are summable under no less restrictive conditions than would insure convergence itself. But by using the same formula of order 2, or 1 higher, various new and interesting results follow. The manner in which the summable properties of the Legendre developments thus lie intermediate between the range of the formulas of orders 1 and 2 has led, it may be added, to a generalization of the whole conception of the formula to include fractional or even incommensurable orders of summation. Out of such a generalization there arise interesting special studies analogous to what we find in the ordinary elementary study of series. Thus, just as we know that in the case of the series

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$$

convergence merges into divergence as p passes through the value 1 from above, so we are able to determine for divergent series just where the critical order, or orders, are at which new information begins to be realized. Furthermore, the extended study of sum formulas simply upon their own merits, that is, without stressing their logical interrelations, has led very naturally to the notion of uniform summability corresponding to that of the uniform convergence of a series. And again, alongside of the same studies, corresponding studies have naturally arisen for divergent integrals. Here as before the underlying idea is that of setting up a formula which shall give the value of all convergent improper integrals and at the same time preserve a meaning and thus assign a value to some integrals that are divergent. As the formulas pertaining to such

studies are not as familiar as those for series, it may be proper to note that if the type integral be taken as

$$I = \int_a^\infty f(\beta)d\beta,$$

then the formula analogous to the Cesàro formula of order 1 and giving the value of I even in some cases of divergence is

$$I = \lim_{x=\infty} \frac{1}{x} \int_a^x d\alpha \int_a^x f(\beta)d\beta,$$

this formula having been first obtained, I believe, by Professor C. N. Moore in his thesis. It is equivalent to the somewhat simpler form

$$I = \lim_{x=\infty} \int_a^x f(\beta)\left(1 - \frac{\beta}{x}\right)d\beta.$$

I shall not attempt any further details concerning sum formulas and the facts derivable either directly or by suggestion from them. We can certainly say, however, that this class of studies, as carried out independently for the various formulas, has greatly increased the range of interest in series in general and has enabled us to see convergent series in particular from new and very instructive points of view.

The second reason alluded to above for the prevalent interest in divergent series despite the lack as yet of any universally accepted general theory about them brings us to a certain very important aspect of the whole which we have not as yet mentioned—an aspect, it may be added, which was entirely disposed of in 11 lines of the encyclopedia article mentioned at the beginning. We refer to what is known as "asymptotic series." This is in reality the oldest aspect which our subject presents. It may be said to have originated in an isolated note by Cauchy in 1843 relating to the well-known series of Stirling

$$\log \Gamma(x) = \tfrac{1}{2}\log 2\pi + (x - \tfrac{1}{2})\log x - x + \frac{B_1}{1\cdot 2}\frac{1}{x} - \frac{B_2}{3\cdot 4}\frac{1}{x^3} + \frac{B_3}{5\cdot 6}\frac{1}{x^5} - \cdots \tag{1}$$

(B_m = mth Bernoulli number).

Cauchy pointed out that this series, though divergent for all

values of x, may be used in computing $\log \Gamma(x)$ when x is large (and positive). In fact, he showed that, having fixed the number n of terms taken, the absolute error committed by stopping the summation at the nth term is less than the absolute value of the next succeeding term, and hence becomes arbitrarily small ($n > 3$) as x increases indefinitely. Cauchy's work on divergent series was confined, however, to the single series (1) and, owing to the overemphasis placed upon convergent processes by the successors of Cauchy and Abel, as mentioned earlier, no further progress was made in this field until the subject at last reappeared after more than forty years in connection with the researches of Poincaré upon the irregular solutions of linear differential equations. Poincaré considered those divergent series (normal series) of the form

$$e^{f(x)}x^{\rho}(A_0 + A_1/x + A_2/x^2 + \cdots); \tag{2}$$

$$f(x) = \text{a polynomial in } x, \qquad \rho = \text{a constant},$$

which for some time had been known to satisfy *formally* homogeneous linear differential equations of certain types having the point $x = \infty$ as a so-called "irregular point," and he showed essentially that in general to every such formal solution there corresponds an *actual* solution which can be represented by (2) in much the same sense as (1) was described above as representing $\log \Gamma(x)$. In view of the important significance of such results both from the standpoint of the possible use of divergent series and from that of the theory of differential equations, Poincaré set apart and discussed in some detail a broad class of divergent series of the special form (2), applying to them the name of "asymptotic series." Poincaré's results, however, in so far as they concerned differential equations, were noticeably incomplete, being limited by certain unfortunate restrictions, and thus his original studies have given rise in later years to numerous researches in which noteworthy advances have been made, though open questions in this connection still remain. Corresponding investigations, likewise begun by Poincaré, pertaining to linear difference equations have also been undertaken in recent years and carried to an advanced stage. Meanwhile another important aspect of the theory of asymptotic series has come into view; namely, that of actually determining the asymptotic developments of any given function—a problem of decided interest

for the study and classification of functions in general. I can perhaps employ the next few moments to no greater advantage than in sketching somewhat more completely these various fundamental problems relative to asymptotic series, pointing out incidentally certain unsolved special problems of noteworthy interest.

Referring again to differential equations, the studies in question may be said to center about the linear homogeneous differential equation

$$(3)\quad y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_n(x)y = 0,$$

wherein the coefficients $a_1(x)$, $a_2(x)$, $\cdots$, $a_n(x)$ are, in the simplest case, rational functions of x and, in the more extended case, are supposed to be developable in series of the form

$$(4)\quad a_r(x) = x^{rk}\left[a_{r,0} + \frac{a_{r,1}}{x} + \frac{a_{r,2}}{x^2} + \cdots\right] \quad (r = 1, 2, \cdots, n),$$

k being zero or a positive integer. In such a differential equation the point $x = \infty$ is in general an irregular point, so that the usual normal solutions are divergent series of the form (2). With reference to these solutions, we may now cite the following fundamental theorem:

"If for the equation (3) the roots m_1, m_2, m_3, $\cdots$, m_n of the so-called characteristic equation, i. e., of the algebraic equation

$$(5)\qquad m^n + a_{1,0}m^{n-1} + a_{2,0}m^{n-2} + \cdots + a_{n,0} = 0,$$

are distinct, equation (3) possesses n linearly independent solutions $y_1, y_2, y_3, \ldots, y_n$ which, for large values of x, are developable asymptotically in the form (2), wherein $f(x)$ is of degree $k + 1$, and $a_0 = 1$; that is, we have

$$(6)\quad y_r \sim e^{f_r(x)}x^{\rho_r}\left[1 + \frac{A_{1,r}}{x} + \frac{A_{2,r}}{x^2} + \cdots\right] \quad (r = 1, 2, 3, \cdots, n),$$

where $f_r(x)$ is a polynomial of degree $k + 1$ in x, while ρ_r is a constant."

If in this theorem the restriction be removed that the roots of the characteristic equation be distinct; that is, if multiple roots are present, the theorem fails and we at once encounter a problem for which no general solution has yet been obtained. Moreover, the theorem as just stated carries with it the assump-

tion that x is real. When x is regarded as complex, much the same results follow, the forms (6) holding now over certain sectors of the plane emanating from the origin, but here again much remains to be investigated, the existing theorems covering only what may be described as the simplest cases.

Contrasted with the same theorem for differential equations, is the corresponding fundamental theorem for linear difference equations:

"Given a homogeneous linear difference equation of the nth order, which we may write in the form

$$y(x+h)+a_1(x)y(x+h-1)+a_2(x)y(x+h-2)+\cdots+a_n(x)y(x)=0, \tag{7}$$

and let it be assumed that the coefficients $a_1, a_2, \cdots, a_n$ are either rational functions of x or are developable in series of the form (4). Then, if the roots $m_1, m_2, \cdots, m_n$ of the characteristic equation (5) are distinct and no one of them equal to zero, equation (7) possesses n linearly independent solutions $y_1, y_2, \cdots, y_n$ valid for large positive values of x and developable asymptotically in the forms

$$y_r \sim [\Gamma(x+1)]^k m_r{}^x x^{\rho_r}\left[1+\frac{A_{1,r}}{x}+\frac{A_{2,r}}{x^2}+\cdots\right] \tag{8}$$

$$(r=1, 2, 3, \cdots, n).$$

In case the characteristic equation presents multiple roots, or a zero root, no corresponding results appear to have been obtained, at least in general, though this whole subject has been interestingly discussed from an altogether different point of view and in a considerably larger measure of completeness by the introduction throughout of the so-called faculty series instead of the usual power series forms. Here again much remains to be done for the case of a complex variable, though a beginning corresponding to that cited above for differential equations has been made. The importance of these studies, both as regards differential and difference equations, lies, of course, in the fact that it is equations of these particular types that play a most fundamental rôle in analysis, both from the function theoretic standpoint and from that of applications. We shall not enter, however, into further details in this direction more than to mention the fact that corresponding studies

for non-homogeneous differential and difference equations have been considered but, like the homogeneous cases, are in but a limited state of completion.

As regards the problem of determining the asymptotic developments of a given function, which I also mentioned a moment ago, the meaning of this class of studies may perhaps be best understood from one or two simple illustrations. Let us take, for example, the following power series in which x is regarded as taking complex as well as real values:

$$(9) \quad f(x) = \sum_{n=0}^{\infty} \frac{x^n}{\rho - n}; \quad \rho = \text{any non-integral constant.}$$

The radius of convergence of this series is easily seen to be 1, so that the series itself yields no information as to the nature of the function $f(x)$ defined by it in the more distant portions of the plane. In order to secure such information and thus be able to follow the course of the function for values of x of large modulus it becomes necessary to develop $f(x)$ in some manner about the point infinity, as for example in power series in $1/x$, but the simple knowledge of the formula for the nth term of the given series (9) provides no immediate way of determining the coefficients of such a development. When once obtained, moreover, it may either converge or it may represent $f(x)$ only asymptotically. How actually to determine the development, whatever be its ultimate character, is the problem, and in the case before us it may be stated that it becomes

$$f(x) \sim -\frac{\pi(-x)^{\rho}}{\sin \pi\rho} - \frac{1}{(\rho+1)x} - \frac{2}{(\rho+2)x} - \cdots,$$

this form holding at least so long as we confine ourselves to any sector of the plane which does not contain the positive half of the real axis. More generally, it may be shown that if we have any power series

$$\sum_{n=0}^{\infty} g(n)x^n,$$

wherein the coefficient $g(n)$ may be regarded as a function of a complex variable n and as such is analytic throughout the entire n plane except for a finite number p of poles, and at the same time, when considered for values of n of sufficiently large modulus, remains less than a constant, then the function $f(x)$

defined by this series will be developable in general throughout the distant portions of the plane either asymptotically or in a convergent series of the specific form

$$f(x) \sim -\sum_{m=1}^{p} r_m - \frac{g(-1)}{x} - \frac{g(-2)}{x^2} - \frac{g(-3)}{x^3} - \cdots,$$

where r_m represents the residue of the function $\frac{\pi g(n)(-z)^n}{\sin \pi n}$ at the rth pole of $g(n)$. Corresponding results for various other type forms of power series have likewise been obtained, and again, similar studies have been extensively carried out for functions defined not by power series, but by altogether different though very important forms, such as infinite products, or faculty series. Sufficient has been said, I judge, on this aspect of asymptotic series so that you will perceive its bearing not only upon the determination of the values of a function in distant regions but also upon the broader problem of the classification of functions in general, since functions may clearly be distinguished from one another in classes corresponding to the different characters of their asymptotic developments. Very much remains to be done in this entire field of investigation.

What we have thus far said may be briefly summarized in the statement that the modern theory of divergent series contains essentially two branches, the first concerning the question as to how a sum may be assigned to a divergent series in general, and the second pertaining merely to the functional properties of that important special class of divergent power series known as asymptotic series. Of these two branches, the second, though characterized by theorems and results which usually bear a high degree of complexity, presents no logical inconsistencies and is thus in quite as satisfactory a state as convergent series themselves, while the first, or problem of summation, when considered as a whole is still in an unsatisfactory logical state because, as pointed out earlier, we have a large variety of sum formulas which, though agreeing with one another when applied to convergent series, fail to do so to a greater or less degree when applied to divergent series. However, all that remains in order to bring about perfect agreement everywhere is, as was also stated earlier, that we place proper limitations either upon the kind of diver-

gent series to be considered, or upon the kind of sum formulas to be retained, or both. In the time that remains to me I may therefore return for a few moments to the question as to how a logically consistent general theory of summation, if there is to be one, may well be constructed. It may be that the limitations which I am about to place may seem too restrictive to some, yet I take for granted that everyone shares with me the instinctive feeling that there should be a logically sound and fairly useful general theory of *some* sort and it is mainly in that spirit that my suggestions will be made.

In approaching the question let us first cast a glance back over the historical genesis of all the various sum formulas, for this is at once suggestive. The earliest and simplest of them is the one growing out of certain studies of Frobenius in 1880 relative to the behavior of the power series

$$\sum_{n=0}^{\infty} a_n x^n \tag{10}$$

for values of x *upon* its circle of convergence. His theorem in substance was as follows: "Suppose that the radius of convergence of (10) is 1, and let $s_n = a_0 + a_1 + a_2 + \cdots + a_n$. Then, we shall have

$$\lim_{x=1-0} \sum_{n=0}^{\infty} a_n x^n = \lim_{n=\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} \tag{11}$$

whenever the limit on the right exists." This is a straightforward result in the theory of functions having at first sight no relation to divergent series, nor indeed did it come to play any recognized part in the development of the latter for a considerable time. When it did it was because the left member of (11) is known to be identical with

$$\sum_{n=0}^{\infty} a_n \tag{12}$$

whenever this series is *convergent*, so it seemed natural in case it was divergent to continue assigning sums s to it in accordance with the formula

$$s = \lim_{n=\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} \tag{13}$$

so long as the limit on the right exists. This particular for-

mula for s found additional justification in the fact that the value it assigns to (12) when divergent is useful in the sense that it is the value of the function $f(x)$ defined by (10) at the point $x = 1$; that is, formula (13) furnishes the analytic continuation of the power series (10) at the point $x = 1$ upon its circle of convergence. It was essentially in this same spirit of useful as well as possible extension of idea that the sum formulas of Cesàro and Hölder, both of which contain (13) as a special case, were set up, as likewise the later transcendental sum formula of Borel wherein a definite integral is involved. These considerations immediately suggest that a logically coherent and at the same time useful theory of summability may be established by limiting ourselves throughout to those series (12) for which the corresponding power series (10) has a non-vanishing radius of convergence and furthermore limiting our use of sum formulas to those which, like the familiar ones of Cesàro and Borel, assign to a divergent series (12) a sum s which is equal to the value of the analytic continuation of this same power series (10) at the point $x = 1$. Such a value for s allows of no duplicity and is therefore unique, thus removing the primary logical defect heretofore mentioned in the present day aspect of the theory. Moreover, a theory thus limited in scope at once satisfies our desideratum of being useful, for it attaches itself in a most fruitful way to the very important subject of analytic continuation in the theory of functions of a complex variable. For example, it may be shown that in general any divergent power series is summable by Cesàro's formula at points upon its circle of convergence, thus furnishing the analytic continuation of the corresponding function at such points, but that it is not summable by this formula at points outside the same circle. On the other hand, if Borel's integral formula be used on the same series, it gives a sum and hence the analytic continuation not only for points upon the circle of convergence, but in certain regions lying outside this circle; namely, within the so-called polygon of summability formed by tangents to the circle at those points which are singular points of the function defined by the corresponding power series. Moreover, in a theory as thus restricted the usual rules for the manipulation and combination of convergent series are in large measure preserved. In short, we have left, it seems to me, a sufficient body of doctrine to be worthy of the name "general theory of summation."

I trust that I have now made clear my own feelings regarding the three questions raised at the outset; first, as to why divergent series have come into such prominence since the appearance of the early volumes of the encyclopedia, second, what has been done that really constitutes a vital advance and third, as to whether such series are at last upon a truly scientific basis. My only fear is that in attempting to couch the whole in very simple form I may have gone too far in this direction and thus violated a principle which, I believe it is said, the poet Browning always carefully observed; namely, of never using so simple a style that the intelligence of one's readers or hearers may be offended. But this is a rather treacherous principle, as most people discover in attempting to read Browning, so I may perhaps be pardoned if I have seemed to depart too far from it.

SOLUTIONS OF DIFFERENTIAL EQUATIONS AS FUNCTIONS OF THE CONSTANTS OF INTEGRATION.

BY PROFESSOR GILBERT AMES BLISS.

(Read before the American Mathematical Society December 29, 1917.)

THE purpose of this note is to prove the differentiability of the solutions of a system of differential equations with respect to the constants of integration by a method which seems more natural and simpler than those which have hitherto been published. Incidentally a restatement of the so-called "imbedding theorem" for differential equations is given, a theorem which is frequently applied in the calculus of variations, and which has been useful, and could be made still more so, in many other connections. It is analogous to the fundamental theorem for implicit functions in its statement that a solution of a system of differential equations given in advance is always a member of a continuous family of such solutions.

Let C be an arc

$$(C) \qquad x = u(\tau), \qquad \tau_1 \leqq \tau \leqq \tau_2,$$

for which the function u is single-valued and has a continuous first derivative. The neighborhood C_ϵ of this arc is the totality of points (τ, x) which satisfy the inequalities

$$(C_\epsilon) \qquad \tau_1 \leqq \tau \leqq \tau_2, \qquad |x - u(\tau)| \leqq \epsilon.$$

In the differential equation

$$(1) \qquad \frac{dx}{d\tau} = f(\tau, x)$$

the function f is supposed

(a) to be single-valued and continuous in C_ϵ;*

(b) to satisfy the Lipschitz condition

$$(2) \qquad |f(\tau, x) - f(\tau, x')| \leqq \kappa |x - x'|$$

whenever (τ, x) and (τ, x') are both in C_ϵ;

(c) to be such that the equation (1) has the arc C as a solution.

By a solution of equation (1) is always meant an arc of the type of C having a continuous derivative and satisfying the equation. The theorem to be proved is then the following:

For every neighborhood C_ϵ of the arc C with the properties just described there exists a second neighborhood C_δ through every point (τ_0, x_0) of which passes one and but one solution of equation (1), *defined and in C_ϵ on the whole interval $\tau_1\ \tau_2$. The function*

$$x = v(\tau, \tau_0, x_0)$$

representing these solutions is continuous and has a continuous derivative $\partial v/\partial\tau$ in the region R of points (τ, τ_0, x_0) satisfying the conditions

$$(R) \qquad \tau_1 \leqq \tau \leqq \tau_2, \qquad (\tau_0, x_0) \text{ interior to } C_\delta.$$

If $f(\tau, x)$ has continuous partial derivatives of the n-th order in C_ϵ, then v and $\partial v/\partial\tau$ also have continuous partial derivatives up to and including those of order n when (τ, τ_0, x_0) lies in the region R.

The existence and continuity of the function $v(\tau, \tau_0, x_0)$ have been established in various ways. For the sake of completeness a proof will be given here which is based upon the method of approximation of Picard, and which makes use of the sequence of functions $\{v_m\}$ defined by the following equations:

* If only the continuity of f with respect to τ is presupposed, then with the help of (b) it is provable that f is continuous in τ and x together. The proof for boundary points of C_ϵ is less direct than for interior points.

$$
\begin{aligned}
v_0(\tau, \tau_0, x_0) &= x_0 + u(\tau) - u(\tau_0), \\
v_1(\tau, \tau_0, x_0) &= x_0 + \int_{\tau_0}^{\tau} f(\tau, v_0)d\tau, \\
&\cdots\cdots\cdots\cdots\cdots \\
v_{m+1}(\tau, \tau_0, x_0) &= x_0 + \int_{\tau_0}^{\tau} f(\tau, v_m)d\tau.
\end{aligned} \tag{3}
$$

To justify the use of these formulas it must first be shown that every function v_m defines points (τ, v_m) in the region C_ϵ where the continuity properties of f are presupposed, provided that the region R in which the values (τ, τ_0, x_0) range is sufficiently restricted. For this purpose select δ so small that in R the relation

$$|v_0 - u| = |x_0 - u(\tau_0)| < \epsilon e^{-\kappa\lambda} = \rho \tag{4}$$

holds, where λ is the length of the interval $\tau_1\tau_2$ and ρ merely a notation for the constant $\epsilon e^{-\kappa\lambda}$. Since on the arc C

$$u(\tau) = u(\tau_0) + \int_{\tau_0}^{\tau} f(\tau, u)d\tau$$

it follows from (3), (4), and (2) that

$$
\begin{aligned}
|v_1 - u| &= \left| x_0 - u(\tau_0) + \int_{\tau_0}^{\tau} [f(\tau, v_0) - f(\tau, u)]d\tau \right| \\
&< \rho\left\{1 + \frac{\kappa|\tau - \tau_0|}{1!}\right\},
\end{aligned}
$$

and hence by a simple induction that for every value of τ on the interval $\tau_1\tau_2$

$$
\begin{aligned}
|v_m - u| &< \rho\left\{1 + \frac{\kappa|\tau - \tau_0|}{1!} + \cdots + \frac{\kappa^m|\tau - \tau_0|^m}{m!}\right\} \\
&\leqq \rho e^{\kappa\lambda} = \epsilon.
\end{aligned}
$$

Hence every function v_m determines an arc in C_ϵ and can be used in the last of formulas (3) to define v_{m+1}.

Furthermore the sequence $\{v_m(\tau, \tau_0, x_0)\}$ of functions continuous in the region R converges uniformly, and its limit function $v(\tau, \tau_0, x_0)$ is therefore continuous. For from (3) and (2)

$$|v_2 - v_1| = \left| \int_{\tau_0}^{\tau} [f(\tau, v_1) - f(\tau, v_0)] d\tau \right|,$$

$$\leqq 2\epsilon \frac{\kappa |\tau - \tau_0|}{1!}$$

since v_0 and v_1 both define curves in C_ϵ and hence differ by at most 2ϵ. An induction as before now gives

$$|v_{m+1} - v_m| \leqq 2\epsilon \frac{\kappa^m |\tau - \tau_0|^m}{m!} \leqq 2\epsilon \frac{\kappa^m \lambda^m}{m!},$$

showing that the series

$$v_1 + (v_2 - v_1) + \cdots + (v_{m+1} - v_m) + \cdots \tag{5}$$

converges uniformly.

From the last of the equations (3) it follows as usual that the limit function $v(\tau, \tau_0, x_0)$ satisfies the equation

$$v(\tau, \tau_0, x_0) = x_0 + \int_{\tau_0}^{\tau} f[\tau, v(\tau, \tau_0, x_0)] d\tau$$

and hence also the differential equation (1). If there were a second solution $w(t)$ of equation (1) through the point (τ_0, x_0) it would satisfy an equation similar to the last one, and consequently also the inequality

$$|w - v| \leqq \left| \int_{\tau_0}^{\tau} \kappa |w - v| \, d\tau \right| \leqq 2\epsilon \frac{\kappa |\tau - \tau_0|}{1!}$$

since 2ϵ is greater than $|w - v|$ on the interval $\tau_1 \tau_2$. But successive application of this relation gives

$$|w - v| \leqq 2\epsilon \frac{\kappa^m |\tau - \tau_0|^m}{m!},$$

which can be true only if v and w are identical, since the second member has the limit zero as m increases.

The proofs of the preceding paragraphs establish the existence, continuity, and uniqueness of the function $v(\tau, \tau_0, x_0)$, and also of its derivative $\partial v / \partial \tau$ since v satisfies (1). To prove that it has the further derivatives described in the theorem, suppose first that $f(\tau, x)$ has continuous first derivatives in C_ϵ. Then each of the functions v_m has a derivative

$$\frac{\partial v_m}{\partial x_0} = 1 + \int_{\tau_0}^{\tau} f_x(\tau, v_{m-1}) \frac{\partial v_{m-1}}{\partial x_0} d\tau \tag{6}$$

and it is proposed to show that the sequence $\{\partial v_m/\partial x_0\}$ converges uniformly for values (τ, τ_0, x_0) in the region R. A well-known theorem* concerning the differentiation of a series term by term, applied to the series (5), then establishes the fact that the limit $v(\tau, \tau_0, x_0)$ of the sequence $\{v_m\}$ has as its derivative the limit of the sequence $\{\partial v_m/\partial x_0\}$.

Consider the two sums

$$(7) \quad B = 1 + \int_{\tau_0}^{\tau} f_x d\tau + \cdots + \int_{\tau_0}^{\tau} f_x \cdots \int_{\tau_0}^{\tau} f_x d\tau + \cdots,$$

$$(8) \quad \frac{\partial v_m}{\partial x_0} = 1 + \int_{\tau_0}^{\tau} A_{m-1} d\tau + \cdots + \int_{\tau_0}^{\tau} A_{m-1} \cdots \int_{\tau_0}^{\tau} A_0 d\tau^m,$$

where $f_x = f_x(\tau, v)$ represents the first partial derivative of f with respect to x, and $A_n = f_x(\tau, v_n)$. The first sum is an infinite series, while the second has a finite number of terms found by successive application of formula (6). A simple inductive proof shows that each term of (8) approaches uniformly the corresponding term of (7) as a limit when m increases indefinitely, since the elements of the sequence $\{v_m\}$ approach uniformly the limit v. Further each term of both sums has absolute value less than the corresponding term of the series

$$1 + \frac{\kappa|\tau - \tau_0|}{1!} + \cdots + \frac{\kappa^n|\tau - \tau_0|^n}{n!} + \cdots$$

or of the series

$$(9) \quad 1 + \frac{\kappa\lambda}{1!} + \cdots + \frac{\kappa^n\lambda^n}{n!} + \cdots,$$

where κ is now the maximum of the absolute value of f_x in C_ϵ. It follows that an integer n can be taken so large that all the terms of the series (7) after the nth have a sum with absolute value less than $\epsilon/3$, and also so large that the same is true for the expression (8). Since the individual terms of (8) approach uniformly those of (7) it is also true that for sufficiently large values of m the sums of the first n terms of (8) and (7) differ by less than $\epsilon/3$. Hence for such values of m the difference between $\partial v_m/\partial x_0$ and the sum of the series (7) is less than ϵ, and the derivatives $\partial v_m/\partial x_0$ approach the sum of the series (7) uniformly for all values (τ, τ_0, x_0) in the region R. The function v has therefore a derivative $\partial v/\partial x_0$ given by the series

* Goursat-Hedrick, A Course in Mathematical Analysis, vol. 1, p. 365.

$$\frac{\partial v}{\partial x_0} = 1 + \int_{\tau_0}^{\tau} f_x(\tau, v)d\tau + \cdots + \int_{\tau_0}^{\tau} f_x(\tau, v) \cdots \int_{\tau_0}^{\tau} f_x(\tau, v)d\tau^m + \cdots, \tag{10}$$

which converges uniformly, since its terms are respectively less in absolute value than those of (9), and $\partial v/\partial x_0$ is therefore continuous in the region R.

Similar reasoning shows the existence and continuity of the derivative $\partial v/\partial \tau_0$. For from equations (3)

$$\frac{\partial v_m}{\partial \tau_0} = -f(\tau_0, x_0) + \int_{\tau_0}^{\tau} A_{m-1}\frac{\partial v_{m-1}}{\partial \tau_0}d\tau$$

$$= -f_0 - \int_{\tau_0}^{\tau} A_{m-1}f_0 d\tau - \cdots$$

$$-\int_{\tau_0}^{\tau} A_{m-1}\cdots\int_{\tau_0}^{\tau} A_1 f_0 d\tau^{m-1} - \int_{\tau_0}^{\tau} A_{m-1}\cdots\int_{\tau_0}^{\tau} A_0 u'(\tau_0)d\tau^m,$$

where $f_0 = f(\tau_0, x_0)$. The comparison series analogous to (9) is ν times that series, where ν is the larger of the maxima of $|f(\tau_0, x_0)|$ and $|u'(\tau_0)|$ in R, and the value of the derivative sought is the continuous sum of the uniformly convergent series

$$\frac{\partial v}{\partial \tau_0} = -f(\tau_0, x_0) - \int_{\tau_0}^{\tau} f_x(\tau, v)f(\tau_0, x_0)d\tau - \cdots - \int_{\tau_0}^{\tau} f_x(\tau, v)\cdots\int_{\tau_0}^{\tau} f_x(\tau, v)f_0(\tau_0, x_0)d\tau^m - \cdots. \tag{11}$$

Finally the equation

$$\frac{\partial v}{\partial \tau} = f(\tau, v) \tag{12}$$

shows that $\partial v/\partial \tau$ is continuous and has continuous first partial derivatives in the region R, since f and v have this property in C_ϵ and R, respectively. This completes the proof of the theorem for the case $n = 1$. The corresponding results for an arbitrary value of n are readily established by induction when the theorem for $n = 1$ has been proved for a system of equations instead of a single one, as will be shown below. In concluding the simpler case it is important to note the relation

$$\frac{\partial v}{\partial \tau_0} = -\frac{\partial v}{\partial x_0} f(\tau_0, x_0), \tag{13}$$

which follows readily from (11) and (10).

The proofs of the preceding theorems for a system of differential equations are of the same character as those above, but the equations used in the proofs will be much more complicated unless notations are used for matrices and multipartite numbers. Peano seems to have been the first to define and apply the moduli of multipartite numbers and matrices in proving existence theorems for differential equations, the case which he considered being that of a linear system.* If x denotes the set of numbers $(x', x'', \cdots, x^{(p)})$, and f the set $(f', f'', \cdots, f^{(p)})$, the single equation (1) will represent the system

$$\frac{dx^{(i)}}{dt} = f^{(i)}(t, x', \cdots, x^{(p)}) \quad (i = 1, \cdots, p). \tag{14}$$

By replacing absolute values everywhere by moduli, the modulus of x for example being

$$\operatorname{mod} x = \sqrt{x'^2 + \cdots + x^{(p)2}}, \tag{15}$$

and using simple properties of multipartite numbers and matrices,† the proofs for the system (14) can be carried through quite simply and the notations will be the same, equation for equation, as in the paragraphs above. To facilitate the reinterpretation of the equations the following table of the symbols used is given:

Positive integers: i, j, m, n, p;

Scalars: $\delta, \epsilon, \kappa, \lambda, \mu, \nu, \rho, \sigma, \tau, e$;

p-partite numbers: $a, f, x, y, z, u, v, w, \partial v/\partial \tau_0, \partial v_m/\partial \tau_0$;

p-square matrices: $A, B, f_x = \partial f/\partial x, \partial v/\partial x_0, \partial v_m/\partial x_0$.

In order to carry through the proofs of the existence and continuity of the derivatives of the p functions v, some additional properties of matrices are needed besides those given in the papers cited above. The definition of the modulus of a matrix given below is equivalent to that of Peano but different in form. It is closely associated with the notion of a limited

* *Mathematische Annalen*, vol. 32 (1888), p. 450.

† See also my paper, "The solutions of differential equations of the first order as functions of their initial values," *Annals of Mathematics*, 2d ser., vol. 6 (1905), p. 58.

matrix used by Hilbert in his theory of quadratic forms in a denumerable infinity of variables, and for the special case here considered is identical with the modulus as defined by E. H. Moore for the purposes of his General Analysis. It is applied here only to square matrices, but would be equally effective for those which are of unequal dimensions, and in particular for a matrix of one row and p columns it gives precisely the value (15). The table of properties in the next paragraph but one contains besides those of Peano some further properties which are useful in the present paper.

Let A be a p-square matrix, and let mod A be the maximum of the bilinear form $Ay \cdot z^*$ on the set of values of y and z which satisfy the equations mod y = mod z = 1. This maximum is attained since the set over which y and z range is closed, and from the elementary theory of maxima and minima the values of y and z which determine the maximum satisfy the linear equations

$$Ay = \rho z, \quad \bar{A}z = \sigma y, \tag{16}$$

from which also

$$\bar{A}Ay = \rho\sigma y, \quad \bar{A}Az = \rho\sigma z, \tag{17}$$

where ρ, σ are scalars and $\bar{A}$ is the matrix formed from A by interchanging rows and columns. From the equations (16) one deduces readily

$$Ay \cdot z = \rho, \quad \bar{A}z \cdot y = Ay \cdot z = \sigma, \tag{18}$$

when mod y = mod z = 1, so that ρ and σ are both equal to the value of the modulus of A. Furthermore from the equations (16) and (18)

$$Ay \cdot Ay = \bar{A}Ay \cdot y = \rho^2,$$

which suggests that the modulus defined above may be identical with that of Peano, the value which he used being the square root of the maximum of $Ay \cdot Ay$ on the set of values y satisfying mod y = 1. By reasoning quite similar to that just used, and with the help of equations (17), this proves to be the case.

The modulus of a matrix has the following properties, which are readily provable:

* The expression $y \cdot z$ is the sum of the products of the elements of y and z; Ay is the m-partite number whose elements are the dot-products of the rows of A by y. For the use of the dot in this connection see Gibbs-Wilson, Vector Analysis, p. 55.

(1) mod A = maximum of $Ay\cdot z$ on the set of values y, z satisfying mod y = mod z = 1,
= maximum of $\sqrt{Ay\cdot Ay}$ on the set of values y satisfying mod y = 1;
(2) every element of A has absolute value $\leqq$ mod A;
(3) mod $A < p^2\delta$ if each element of A has absolute value less than δ;
(4) mod $\kappa A = |\kappa|$ mod A;
mod $Ay \leqq$ mod A mod y;
mod $AB \leqq$ mod A mod B;
(5) mod $(A + B) \leqq$ mod $A +$ mod B,
mod $(A - B) \geqq |$mod $A -$ mod $B|$;
(6) mod A is a continuous function of the elements of A;
(7) mod $\int_{\tau_0}^{\tau_1} A d\tau \leqq \left|\int_{\tau_0}^{\tau_1} \text{mod } A d\tau\right|$;
(8) $|\lambda| <$ mod $(dA/d\tau)$ for every derivative number λ of mod A.

In the statement (7) the elements of A are supposed to be integrable functions of τ on the interval $\tau_0 \leqq \tau \leqq \tau_1$, and in (8) they are supposed to have unique derivatives at the value of τ considered. The property (8) is not necessary for the applications of the matrix theory in the present paper.

The property (2) holds because, by choosing all elements of y and z to be zero except one element equal to $+1$ or -1 in each, the value $Ay\cdot z$ may be made equal to the absolute value of a selected one of the elements of A. In the expression $Ay\cdot z$ with mod y = mod z = 1 there are p^2 terms each consisting of an element of A multiplied by two numbers with absolute values $\leqq 1$. Hence when each term of A has absolute value less than δ the maximum of $Ay\cdot z$ surely does not exceed $p^2\delta$. The proof of the first formula (4) is immediate. The second part follows since for every y

$$\sqrt{Ay\cdot Ay} = \sqrt{\frac{Ay\cdot Ay}{(\text{mod } y)^2}}\ \text{mod } y \leqq \text{mod } A \text{ mod } y.$$

To prove the third part of (4), suppose that y, z are sets of values having mod y = mod z = 1 and giving $ABy\cdot z$ its maximum. Then

$$\text{mod } AB = ABy\cdot z = \frac{ABy\cdot z}{\sqrt{By\cdot By}}\ \sqrt{By\cdot By} \leqq \text{mod } A \text{ mod } B.$$

If y and z are similarly the values giving $(A + B)y\cdot z$ its maximum, then

$$\begin{aligned}\text{mod } (A+B) &= (A+B)y\cdot z \\ &= Ay\cdot z + By\cdot z \leqq \text{mod } A + \text{mod } B.\end{aligned}$$

The second part of (5) follows from the first since

$$\text{mod } A \leqq \text{mod } (A-B) + \text{mod } B,$$

$$\text{mod } B \leqq \text{mod } (B-A) + \text{mod } A.$$

If $A + \Delta A$ is a matrix formed from A by giving its elements increments with absolute values less than δ, then from the second property (5), and (3),

$$|\text{mod } (A+\Delta A) - \text{mod } A| \leqq \text{mod } \Delta A < p^2\delta,$$

which proves the statement (6). The property (7) is readily proved by applying the first formula (5) to the sum whose limit is the definite integral on the left, and then taking the limit. Finally the inequality

$$\left|\frac{\text{mod } (A+\Delta A) - \text{mod } A}{\Delta\tau}\right| \leqq \text{mod } \frac{\Delta A}{\Delta\tau}$$

is deducible with the help of the second property (5), and it justifies (8) since every derivative number of mod A is the limit of the fraction on the left when $\Delta\tau$ approaches zero over a suitably selected sequence of values.

In the equation (6) for the system (14) the symbol $\partial v_m/\partial x_0$ represents the matrix of derivatives $\| \partial v_m^{(i)}/\partial x_0^{(j)} \|$ $(i, j = 1, \cdots, p)$ and the first term on the right is to be interpreted as the identity matrix I. A similar agreement holds for the first terms of (7), (8), and (10). The symbol f_x now represents the matrix of derivatives $\| \partial f^{(i)}/\partial x^{(j)} \|$. Hence in (6) and the three last mentioned equations every term is a p-square matrix. With the help of the matrix properties (7) and (4_2) above

$$\begin{aligned}&\text{mod} \int_{\tau_0}^{\tau} A_{m-1}\cdots\int_{\tau_0}^{\tau} A_{m-n}d\tau^n \\ &\quad\leqq \left|\int_{\tau_0}^{\tau} \text{mod } A_{m-1}\cdots\int_{\tau_0}^{\tau} \text{mod } A_{m-n}d\tau^n\right| \leqq \frac{\kappa^n|\tau-\tau_0|^n}{n!},\end{aligned}$$

where κ is the maximum of mod f_x in the region C_ϵ. Hence the terms of the series (9) exceed the moduli of the respective

terms in (7) and (8). An integer n can now be taken so large that the modulus of the sum of the terms of (8) after the nth is less than $\epsilon/3$, and so that the same is true for the sum of the terms after the nth in the series (7). Since the elements of the sequence $\{v_m\}$ converge uniformly to the elements of v, the elements of the matrix represented by the first n terms of the sum (8) converge to the corresponding elements of the similar matrix from the series (7), and by an application of the matrix property (3) it is clear that the modulus of the difference of these matrices is less than $\epsilon/3$ when $m > n$ is sufficiently large. For such values m therefore

$$\operatorname{mod}\left(\frac{\partial v_m}{\partial x_0} - B\right) < \epsilon,$$

and because of the matrix property (2) each element of $\partial v_m/\partial x_0$ converges uniformly, as m increases, to the respective element of B. Hence every one of the elements of v has a derivative with respect to every element of x_0 and the matrix $\partial v/\partial x_0$ of these derivatives is B. Finally the series of moduli of the terms of (10) converges uniformly, implying also, on account of property (2), the uniform convergence of every series of corresponding elements in the matrices which the terms of (10) represent, and showing therefore that every element of the matrix $\partial v/\partial x_0$ is a continuous function.

The proof of the existence and continuity of the p derivatives $\partial v/\partial \tau_0$ is so similar that it is unnecessary to exhibit it in detail.

The proofs which have been given above establish the properties of the solution $v(\tau, \tau_0, x_0)$ of a system of equations for the case $n = 1$ described in the theorem, and the proof can now be given for the general case by an induction. Suppose that the existence and continuity of the nth derivatives of v and $\partial v/\partial \tau$ have been proved to be a consequence of the assumption that f has continuous nth derivatives, and suppose further that the functions f have continuous derivatives in C_ϵ of order $n + 1$. The system of $2p + 1$ equations

$$(19) \qquad \frac{d\tau_0}{d\tau} = 0, \qquad \frac{dx_0}{d\tau} = 0, \qquad \frac{dw}{d\tau} = f_x(\tau, v)w$$

in the variables τ, τ_0, x_0, w has the solution

$$(\Gamma) \qquad \tau_0 = \text{const.}, \; x_0 = \text{const.}, \; w = \frac{\partial v}{\partial x_0} w_0$$

with the initial values τ_0, x_0, w_0 for $\tau = \tau_0$, as one readily sees by differentiating equation (12) for the elements of x_0 and from the fact that $\partial v/\partial x_0$ is the identity matrix when $\tau = \tau_0$. There is a neighborhood Γ_ϵ in which the second members of equations (19) have continuous nth derivatives, provided that (τ, τ_0, x_0) is in R, since v has this property and f has by hypothesis continuous derivatives of order $n + 1$. Hence there is also a region P for the solution Γ, analogous to R for C, in which the solution Γ has continuous nth derivatives. This shows that all of the elements of the matrix $\partial v/\partial x_0$ have continuous nth derivatives since w_0 is an arbitrary constant multipartite number. The same is true of the elements of the multipartite function $\partial v/\partial \tau_0$ from equations (13), and for the elements of $\partial v/\partial \tau$ from equations (12). Hence all of the derivatives of v of order $n + 1$ exist and are continuous as described in the theorem. That $\partial v/\partial \tau$ has the same property follows at once from the relations (12).

In conclusion it may be remarked that the theorems concerning the dependence of the solutions v upon parameters involved in the equations (14) require no proofs essentially different from those given above. A system of the form

$$\frac{dx}{d\tau} = f(\tau, x, a),$$

where a is a multipartite set of parameters, may in fact be replaced by the system

$$\frac{dx}{d\tau} = f(\tau, x, a), \qquad \frac{da}{d\tau} = 0$$

in the variables τ, x, a. If the functions f have continuous derivatives of order n in τ, x, a, then the theorem proved above justifies the statement that the derivatives of order n for the solutions $x = v(\tau, \tau_0, x, a)$ and their derivatives v_τ will also exist and be continuous in the region analogous to R for these equations.

INVOLUTIONS ON THE RATIONAL CUBIC.

BY PROFESSOR R. M. WINGER.

(Read before the San Francisco Section of the American Mathematical Society October 27, 1917.)

Introduction.

1. THE general subject of involution as applied to rational curves has been widely studied, notably by Weyr, Stahl, Coble and many Italian writers. It is the purpose of this paper to discuss certain involutions on the rational cubic, R^3.

If s_i denote the elementary symmetric functions of coordinates $x_1, x_2, \ldots, x_n$ of n points (elements in the binary domain), the most general involution of order n, $I_{n-1.1}$, i. e., one in which $n-1$ points of a set determine the remaining one, will be defined by

$$a_0 s_n + a_1 s_{n-1} + a_2 s_{n-2} + \cdots + a_{n-1} s_1 + a_n = 0. \tag{1}$$

The involution is thus made up of all sets of n points apolar to a fixed set, the n-fold points of the involution, given by

$$a_0 x^n + \binom{n}{1} a_1 x^{n-1} + \binom{n}{2} a_2 x^{n-2} + \cdots + \binom{n}{n-1} a_{n-1} x + a_n = 0. \tag{2}$$

The following alternative and equivalent definition is serviceable when the n points of a set are represented implicitly by an equation: An $I_{n-1,1}$ is an $(n-1)$-parameter family of binary forms of order n

$$f_0 + k_1 f_1 + k_2 f_2 + \cdots + k_{n-1} f_{n-1}. \tag{3}$$

More generally, if $n-r$ points of a set suffice to determine the remaining r, x_i must satisfy r equations of the type (1) and (3) reduces to an $(n-r)$-parameter family. The corresponding involution is denoted by $I_{n-r,\,r}$.

2. Choosing for triangle of reference the nodal tangents and the line of flexes, the equation of the curve may be written in the canonical form

$$x_1 = 3t^2, \qquad x_2 = 3t, \qquad x_3 = t^3 + 1. \tag{4}$$

The peculiar adaptability of the language of involution to describe the properties of the curve will appear from a few examples. Thus the condition that three points be on a line is

$$s_3 + 1 = 0, \tag{5}$$

i. e., (5) defines the involution which comprises all line sections. If the line is tangent at τ, meeting again at t, the equation becomes

$$\tau^2 t + 1 = 0. \tag{6}$$

Again (from (6)) *the parameters of contacts of tangents from an arbitrary point of R^3 belong to the quadratic involution $s_1 = 0$ whose double points are the nodal parameters.**

Finally, *lines joining pairs of points in a quadratic involution envelop a conic perspective to R^3.*†

The line $\overline{t_1 t_2}$ is

$$(s_1 s_2 - 1)x_1 + (s_1 - s_2^2)x_2 - 3s_2 x_3 = 0. \tag{7}$$

If the double points of the involution are given by

$$at^2 + 2bt + c = 0, \tag{8}$$

the equations of the perspective conics in lines, found by requiring that (8) be apolar to (7) considered as a quadratic in t_2, are

$$u_1 = bt^2 - ct + a, \qquad u_2 = ct^2 - at + b, \qquad u_3 = -3bt. \tag{9}$$

These conics have each three contacts with R^3 which belong to an $I_{2.1}$, $s_3 = 1$, whose triple points are the sextactic points.‡

Among the tri-tangent conics must be counted the degenerate conics consisting of two tangents from a point of the curve. *Thus the point t and the contacts of tangents from t are a set apolar to the sextactic points.* The equation of any composite conic, e. g., the pair of tangents from t_1, is found at once by taking the discriminant of (7) considered as a quadratic in t_2.

Since the triple points are a set in the involution, there is one conic touching at the sextactic points. This is a remarkable conic which we shall call N. It is obtained by requiring

*This theorem, discovered independently, is referred to by Weyr, *Wiener Berichte*, vol. 79 (1879), p. 429 ff., as known.

† Coble, "Symmetric binary forms and involutions," *Amer. Jour. Math.*, vol. 32, p. 358.

‡ The equations of the three sextactic conics are given by (9) when $b = 1; a = c = -3, -3\omega, -3\omega^2$, $(\omega^3 = 1)$.

that the double points (8) of the quadratic involution be the nodal parameters. In this case $a = c = 0$ and the conic is

$$(10)\qquad \begin{aligned} &u_1 = t^2,\ u_2 = 1,\ u_3 = -3t, \quad \text{or} \quad u_3^2 - 9u_1u_2 = 0,\\ &x_1 = 3,\ x_2 = 3t^2,\ x_3 = 2t, \quad \text{or} \quad 9x_3^2 - 4x_1x_2 = 0. \end{aligned}$$

Combining the foregoing discussion with a theorem stated at the beginning of this section, we may say: *The lines joining pairs of contacts of tangents from the points of R^3 envelop a conic N which touches the nodal tangents (where they meet the line of flexes) and has contacts with R^3 at the sextactic points.*

3. As an example of an $I_{1.2}$ may be mentioned the involution set up by the pencil of lines $(ux) + \lambda(vx) = 0$. The lines will cut out a pencil, i. e., an $I_{1.2}$ of binary cubics, say $u + \lambda v$. The contacts of tangents from the center of the pencil which are the double points of the involution are given by the Jacobian J of u and v.*

Are there any lines u whose cubi-covariant points are also line sections u'? If so, these lines may be taken as the base lines of a pencil and the involution becomes $u + \lambda u'$. That is, from the intersection P of u and u' can be drawn not four tangents but a pair of repeated tangents. This can happen if and only if P is the intersection of two flex tangents. Hence if u is a line of a pencil with center P, its cubi-covariant points lie on a line u' of the same pencil. *In other words the locus of lines u whose cubi-covariant points lie on a line u' consists of the vertices of the triangle of flex tangents, while u' envelops the same points.*

Lines u and u' in any pencil P belong to a quadratic involution of lines whose double lines are the two flex tangents meeting at P.

The points P are $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$ and are therefore fully perspective with the reference triangle.

Other Contact Conics.

4. To find the intersections of R^3 with a general conic we substitute equations (4) in the trilinear equation of the conic. The result will be a sextic in t, in which obviously the coefficient of the highest power is the same as the constant term. Hence the necessary and sufficient condition that six points lie on a conic is that their parameters satisfy the equation

* Salmon, Higher Algebra, fourth edition, p. 162.

$$s_6 = 1, \tag{11}$$

or that they belong to a sextic involution $I_{5,1}$.

We shall get contacts when two or more t's come together.* Passing to the extreme case, suppose all six t's coincide. Then

$$t^6 - 1 = (t^3 + 1)(t^3 - 1) = 0, \tag{12}$$

which includes the flexes among the sextactic points. The conics then degenerate of course into the flex tangents repeated.

5. Next suppose five points coincide at τ. Then

$$\tau^5 t - 1 = 0, \tag{13}$$

which says that at each point τ of R^2 there is a conic with 5-point contact,† but through a given point t five such conics pass. Moreover the product s_5 of the five quintactic parameters satisfies the equation $s_5 = 1/t$ (from (13)). Hence, by (11), *the quintactic points of the five quintactic conics which pass through t (simply) lie on a conic with t.*

6. Let four intersections coincide at τ and two at t, or

$$\tau^4 t^2 = 1. \tag{14}$$

There would seem to be four conics with simple contact at t and with 4-point contact elsewhere. But among these are the tangent lines from t each counted twice and therefore only two proper conics with contacts satisfying the equation

$$\tau^2 t = 1. \tag{15}$$

Or the quartactic points τ of conics touching at t are given by

$$\tau^2 - 1/t = 0. \tag{16}$$

Hence they lie on a line with t. Moreover they are harmonic with the contacts of tangents from t. They are likewise harmonic with the nodal parameters; hence the line joining them is a line of conic N and their tangents meet again on the curve, viz., at $-t$. Again $-t$ is on a line with the contacts of tangents from t. These statements apply equally well to $-t$.

* Contact here simply means coincident intersections and will include improper contact as well as ordinary tangency, i. e., coincidence of consecutive parameters.

† We shall call this a *quintactic* point and the conic a quintactic conic, adopting similar terms for the other cases.

We thus have a striking configuration which may be described as follows:

From a point $(-1/t^2)$ *of* R^3 *draw the two tangents* t *and* $-t$ *and from each of these points the pair of tangents. The two pairs of contacts of tangents from* t *and* $-t$ *are harmonic and form therefore with the nodal parameters three mutually harmonic pairs. The line joining contacts of tangents from* t *passes through* $-t$ *and is therefore a tangent from* $-t$ *to conic* N*, and vice versa. The other tangent to* N *from either point is the junction of the two. At either point* (t *or* $-t$) *two conics touch which have quartactic points at contacts of tangents from the other.*

7. If now the six intersections coincide in two triples, the equation becomes

$$t^3\tau^3 = 1, \quad \text{or} \quad \tau^3 - 1/t^3 = 0. \tag{17}$$

Hence at each point t of R^3 there are three osculating conics which osculate the curve again at points $1/t$, ω/t, ω^2/t respectively. At each of these points there are three osculating conics which osculate again, one each at t, ωt and $\omega^2 t$. *We have thus a configuration of six points and nine conics. If the points are arranged in two rows, as*

$$\begin{matrix} t & \omega t & \omega^2 t \\ 1/t & \omega/t & \omega^2/t \end{matrix}$$

then at each point in either row three conics osculate, each of which osculates again at one point of the other row. The six points themselves lie on a tenth conic whose equation is

$$9t^3x_3^2 - (t^3+1)^2\, x_1x_2 = 0.$$

Moreover the two triangles as written are harmonically perspective from each of the flexes.

8. Finally we obtain tri-tangent conics when the points of (11) coincide in pairs. The contacts then satisfy the equation

$$s_3^2 = t_1^2\, t_2^2\, t_3^2 = 1.$$

But if the right side is -1, the conic is any line section repeated. The contacts of all tri-tangent conics (except repeated lines) therefore belong to the $I_{2,1}$, $s_3 = 1$. Since this is the same involution as that defined by the sextactic points we infer that the perspective conics (9) Art. 2 embrace all tri-tangent conics.

Hyperosculating Curves.

9. The foregoing method can be applied to the study of other contact curves of the R^3. We shall restrict our attention, however, to those which have complete intersection at a point. Such curves of order n will be called *hyperosculating* curves, designated by H_n. The contacts, denoted by P_{3n}, will be called hyperosculating points. The simplest of these curves are the flex tangents, of course, which taken n times must be reckoned among all H_n's. Similarly the sextactic conics counted n times will be included among the H_{2n}'s of even degree, etc.

If now the x's from (4) are substituted in a general ternary equation of degree n, there results a binary equation of degree $3n$ in t, wherein the coefficient of the highest power of t differs at most in sign from the constant term. Denoting by s_{3n} the product of the roots of this equation, we have as the condition that $3n$ points be the complete intersections of R^3 and a curve of degree n

$$s_{3n} = (-1)^n. \tag{18}$$

i. e., that they belong to an involution, $I_{3n-1,1}$.

Hence the contacts, P_{3n}, of hyperosculating curves are given by

$$t^{3n} = (-1)^n. \tag{19}$$

10. There will be two cases according as n is odd or even.

Case I. n odd. When $n = 1$ we have the points of inflexion. That is, there are 3 H_1's whose P_3's lie on a line.

When $n = 3$, (19) is

$$t^9 + 1 = (t^3 + 1)(t^6 - t^3 + 1). \tag{20}$$

There are thus 9 H_3's whose P_9's are on a cubic. But three of these P_9's are the flexes, counted three times, and are therefore the complete intersections of a line and R^3. *Hence there are* 6 *proper H_3's whose contacts are on a conic.*

In general, n odd, the contacts of H_n's are given by

$$t^{3n} + 1 = 0. \tag{21}$$

Here $s_{3n} = -1$. Hence there are $3n$ points at which H_n's, including degenerate cases, can be drawn. These points are the complete intersections of R^3 and a C_n. This C_n, however,

is always composite, since it contains the line C_1 of flexes. It may contain also other factors C_{k_i} at whose intersections can be drawn degenerate H_n's which are $H_{n/r}$'s of lower order repeated r times.

If $C_n = C_1C_{k_1}C_{k_2} \ldots C_{k_j}C_{n'}$, it follows that *there are* $3n'$ P_{3n}*'s at which proper* H_n*'s can be drawn and these points are the complete intersections of* R^3 *and* $C_{n'}$. For the lower values of n the C_n's are appended, from which can be inferred the number of P_{3n}'s at which proper H_n's can be drawn; e. g., there are 12 P_{15}'s at which proper H_5's can be drawn and these points are on a quartic C_4.

$$C_3 = C_1C_2, \quad C_5 = C_1C_4, \quad C_7 = C_1C_6, \quad C_9 = C_1C_2C_6, \tag{22}$$
$$C_{11} = C_1C_{10}, \quad C_{13} = C_1C_{12}, \quad C_{15} = C_1C_2C_4C_8.$$

11. *Case* II. n even and equal to $2m$. The hyperosculating points are now given by

$$t^{6m} - 1 = (t^{3m} - 1)(t^{3m} + 1). \tag{23}$$

There are two cases according as m is odd or even.

(*a*) When m is odd the second factor of (23) gives points P_{6m} which are P_{3m} taken twice; while the other factor

$$t^{3m} - 1 = (t^3 - 1)(t^{3m-3} + t^{3m-6} + \ldots + t^3 + 1). \tag{24}$$

The first factor of (24) corresponds to the sextactic points which are to be taken m times. *The other factor indicates that there are* $(3m - 3)$ P_{3n}*'s which lie on a* C_{m-1}.

(*b*) When m is even the first factor of (23) names contacts of $(H_m\text{'s})^2$. *The other factor says that there are* $3m$ P_{3n}*'s which lie on a* C_m. These curves C_{m-1} and C_m may or may not be composite, but as above we can infer that the points at which proper H_n's can be drawn are the complete intersections of R^3 and a $C_{m'}$.

In particular if n is a power of 2, say $n = 2^a$, (19) becomes

$$(t^{3\cdot 2^a} - 1) = (t^{3\cdot 2^{a-1}} - 1)(t^{3\cdot 2^{a-1}} + 1) = 0, \tag{25}$$

the first factor of which names contacts of curves $H_{2^{a-1}}$ repeated, except when $a = 1$. The second factor gives proper $P_{3\cdot 2^a}$'s which by (18) lie on a $C_{2^{a-1}}$. Moreover in virtue of relation (6) these $P_{3\cdot 2^a}$'s are contacts of tangents from proper $P_{3\cdot 2^{a-1}}$'s. We have thus the following chain theorem: *The*

3 sextactic points are contacts of tangents from the flexes P_3. The 6 contacts of tangents from the sextactic points are the points P_{12}. The 12 contacts of tangents from P_{12} in turn are the points P_{24}, and so on ad infinitum.

UNIVERSITY OF OREGON.

RELATED INVARIANTS OF TWO RATIONAL SEXTICS.

BY PROFESSOR J. E. ROWE.

(Read before the American Mathematical Society September 4, 1918.)

LET the parametric equations of the R_3^6, the rational curve of order six in three dimensions, be

$$x_i = \delta^6{}_{it} \equiv a_i t^6 + 6b_i t^5 + 15c_i t^4 + 20d_i t^3 + 15e_i t^2 + 6f_i t + g_i \quad (i = 1, 2, 3, 4), \tag{1}$$

and let the parametric equations of the R_2^6, the rational plane curve of order six, be of the form

$$x_1 = \alpha_t{}^6 \equiv a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6,$$
$$x_2 = \beta_t{}^6 \equiv a' + b't + c't^2 + d't^3 + e't^4 + f't^5 + g't^6,$$
$$x_3 = \gamma_t{}^6 \equiv a'' + b''t + c''t^2 + d''t^3 + e''t^4 + f''t^5 + g''t^6.$$

It is well known that all plane sections of the R_3^6 are apolar to a doubly infinite system of binary sextics, and that all line sections of the R_2^6 are apolar to a triply infinite system of binary sextics. We shall let the four binary sextics $\delta_{it}{}^6$ of (1) be four linearly independent sextics of the apolar system of the R_2^6, and the $\alpha_t{}^6, \beta_t{}^6, \gamma_t{}^6$ of (2) be three linearly independent sextics of the apolar system of the R_3^6. Our purpose is to point out briefly the relation between the invariants of the R_2^6 and the invariants* of the R_3^6.

By means of the twelve equations

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* This relation must not be confused with the correspondence between invariants of the R_2^n and covariant surfaces of the R_3^n.

$$
\begin{aligned}
&a_i a - b_i b + c_i c - d_i d + e_i e - f_i f + g_i g = 0,\\
(3)\qquad &a_i a' - b_i b' + c_i c' - d_i d' + e_i e' - f_i f' + g_i g' = 0,\\
&a_i a'' - b_i b'' + c_i c'' - d_i d'' + e_i e'' - f_i f'' + g_i g'' = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, 3, 4),
\end{aligned}
$$

it may be easily proved that the four-rowed determinants of the matrix of the coefficients of δ_{it}^6 of the type $|abcd|$ are proportional to the complementary three-rowed determinants of the matrix of the coefficients of α_t^6, β_t^6, γ_t^6 of the type $|ef'g''|$. Let T denote the substitution of the three-rowed determinants of (2) for the proportional four-rowed determinants of (1), and T^{-1} the inverse substitution.

Invariants of the R_3^6 are combinants of the four sextics δ_{it}^6, and conversely, and these are rationally expressible in terms of the determinants of the type $|abcd|$. Invariants of the R_2^6 are combinants of α_t^6, β_t^6, γ_t^6, and conversely, and these are rationally expressible in terms of the determinants of the type $|ab'c''|$. The combinants of δ_{it}^6 are implicit invariants of the R_2^6 which become explicit invariants of the R_2^6 after the application of T. Similarly, combinants of α_t^6, β_t^6, γ_t^6 are implicit invariants of the R_3^6 which are transformed into explicit invariants of the R_3^6 by means of T^{-1}. Hence any explicit invariant I of the R_3^6 is transformed into an explicit invariant I' of the R_2^6 by means of T. Similarly, $T^{-1} I' = I$. It is evident that the order of I in the $|abcd|$ is the same as that of I' in the $|ab'c''|$. We shall now mention a few illustrations of this relation.

If U' is the undulation invariant of the R_2^6, $T^{-1} U' = U$ is the stationary line invariant of the R_3^6. From P, the pentatactic plane invariant of the R_3^6, we obtain $TP = P'$, the cusp invariant of the R_2^6. Similarly, from Q, the quinquesecant line invariant of the R_3^6, we derive $TQ = Q'$ whose vanishing defines an R_2^6 such that any six of its collinear points have parameters apolar to a binary quintic. If $N = 0$ is the necessary and sufficient condition that the R_3^6 have a node, $TN = N' = 0$ defines an R_2^6 which has one secant that cuts out a cyclotomic set of parameters.

PENNSYLVANIA STATE COLLEGE,
May, 1918.

IN MEMORIAM: ELLERY WILLIAM DAVIS.

BY PROFESSOR E. R. HEDRICK

(Read at the Chicago meeting of the American Mathematical Society April 12, 1918.)

AT the invitation of the committee on the programme I have the honor to pay brief tribute to the friend and counsellor of many of you, and of mine, our late colleague, Ellery William Davis. It is fitting indeed that this Society pause in the midst of our scientific session to mark the passing of such a life.

He was a mathematician of insight and power. His published works contain many titles of lasting interest. Among those of technical mathematical interest may be noted particularly the following:

1. "An expression of the coordinates of a point on a binodal quartic curve as rational functions of the elliptic functions of a variable parameter." *American Journal of Mathematics*, volume 5 (1882), pages 333–336.
2. "Geometrical illustrations of some theorems in number." *American Journal of Mathematics*, volume 15 (1893), pages 84–90.
3. "A note on the invariance of the factors of composition of a group." *American Journal of Mathematics*, volume 19 (1897), page 191.
4. "On the sign of a determinant's term." *American Journal of Mathematics*, volume 19 (1897), page 383.
5. "Note on special regular reticulations." *Bulletin of the American Mathematical Society*, volume 4 (1897–1898), pages 529–530.
6. "The group of the trigonometric functions." *Bulletin of the American Mathematical Society*, volume 5 (1898–1899), pages 380–381.
7. "Some groups in logic." *Bulletin of the American Mathematical Society*, volume 9 (1902–1903), pages 346–348.

Other papers, less well known, which have appeared in journals not wholly mathematical are:

8. "A definition of mathematical probability." Baldwin's Encyclopedia of Philosophy and Psychology, 1902, pages 344–353.

9. "The elliptic functions and the general symmetric group in four letters." *Nebraska University Studies*, volume 4 (1904), pages 231–247.
10. "The imaginary in geometry." *Nebraska University Studies*, volume 10 (1910), pages 1–58.
11. "Cantor's leap into the transfinite." *Mid-West Quarterly*, volume 4 (1917), pages 239–250.

Among other publications, two of his books have exerted influence on mathematicians and the teaching of mathematics. One of these, "The Logic of Algebra," should find a place in every mathematical library. This he himself regarded as his most important work and it is certainly worthy of a lasting place in mathematical literature.

Finally his text on the Calculus is one which is peculiarly stimulating and will probably affect the teaching of that subject for a long time.

Though this work was good, it alone would not account for the high esteem in which he was held. One lesson to us all is that his gift for friendship, his warm humanism, his broad sympathies, even more than his direct mathematical productiveness, are the bases of that strong feeling of respect and admiration which he so generally inspired. I have seen a number of notices of his passing which I should be glad to read to you, but I shall leave them with the remark that men of all classes have felt what I have expressed in this paragraph.

I might quote from two letters which I received after the programme for this meeting was printed. One of them from a well known woman says: "I am impelled to write to you to say that I count my three years experience in teaching in his department a very rich memory. . . . In particular, I should emphasize his pains in giving a young instructor free scope in whatever field the instructor might be interested. He was unselfish and a very generous spirit." The other is from a man recognized as among the first of American mathematicians. He says: ". . . I feel like saying that few other American mathematicians have encouraged me more by their sympathetic interest than Ellery Williams Davis did."

One illustration of his keenness of vision is now particularly noteworthy, his attitude toward this war and toward Prussianism. Early in the struggle in 1914 he voiced the opinion that this was our war and that the defeat of Prussia was essential to the preservation of liberty and democracy throughout

the world. To us this is significant as a striking instance of his vision and of his unhesitating support of right and high morality regardless of the immediate effects upon his own personal interests. In mathematical circles the same fearlessness of vision has led him to support warmly those projects or innovations which seemed dangerous to the more timorous. I may mention his attitude toward maintaining requirements in mathematics where real grounds exist and for abolishing the requirements otherwise. His attitude toward those reforms which started with the Perry movement and which have since broadened to much wider scope is well known. Finally, his support and sympathetic interest in the Mathematical Association of America is known to all who were interested in that movement.

Ellery Williams Davis was born in Oconomowoc, Wisconsin, on March 29, 1857. He died in Lincoln, Nebraska, Sunday, February 3, 1918. He was graduated with the degree of A.B. at the University of Wisconsin and with the degree of Ph.D. at Johns Hopkins in 1884. On June 20, 1886, he was married to Miss Annie T. Wright, who with four sons and a daughter survives him.

He was professor of mathematics at Florida Agricultural College from 1884 to 1886, at South Carolina College from 1886 to 1893, and at the University of Nebraska from 1893 to 1918. He was also dean of the department of liberal arts at Nebraska from 1901 to 1918.

In closing, let me put into words sentiments that I know are in the hearts of all who knew him. In his death we feel a personal sense of loss and we believe that university spirit and mathematical science in this country, which he so loyally upheld and furthered, have suffered through his passing.

A CORRECTION.

I WISH to call attention to the following errata in my note, "Some theorems of comparison and oscillation" in the April BULLETIN.

(1) On page 331, among the conditions under Theorem I, "$\alpha_2(x)/\alpha_1(x)$ never decreases" should read "$-\alpha_2(x)/\alpha_1(x)$ never decreases."

(2) On page 332, above the middle of the page, the statement "x approaches c" obviously should read "x approaches the first root of y_1 greater than a."

(3) Possibly an explicit statement should be made that $V_i(x)$, $i = 1, 2$, does not vanish identically over any interval, as is shown by the conditions on α_1 and α_2, and that the point l of case I is chosen so that $V_2(l) \neq 0$.

(4) The wording of Theorem I as regards the conditions on K_1, K_2, G_1 and G_2 is careless and could be improved by omitting the word "absolutely" and making no statement as to the integrability of K_1 and K_2.

TOMLINSON FORT.

NOTES.

THE July number (volume 40, number 3) of the *American Journal of Mathematics* contains the following papers: "Interpolation properties of orthogonal sets of solutions of differential equations," by O. D. KELLOGG; "Directed integration," by H. B. PHILLIPS; "P-way determinants, with an application to transvectants," by L. H. RICE; "On a certain general class of functional equations," by W. H. WILSON; "Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order," by R. G. D. RICHARDSON.

THE editors of the *Periodico di Matematica* and its *Supplemento* announce that, on account of the war, the publication of both periodicals will be temporarily discontinued.

AT the meeting of the Edinburgh Mathematical Society held June 14, the following papers were read: By Miss E. PAIRMAN, "Relations connected with generalized differentiation"; by E. T. WHITTAKER, "Some new expansions in series of polynomials."

THE late Professor GASTON DARBOUX left a large part of his library to the newly established reading room of the department of mathematical sciences of the University of Paris.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1918–1919:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Celestial mechanics, three hours.—By Professor L. DONATI: Electrodynamics; electro-optics; relativity, three hours.—By Professor F. ENRIQUES: Foundations of mathematics: (1) geometry, infinitesimal analysis, mechanics and cosmology in ancient Greece; (2) modern critique of principles, three hours.—By Professor S. PINCHERLE: Volterra's and Fréchet's functional calculus; existence theorems for differential equations; integral equations, three hours.

UNIVERSITY OF CATANIA.—By Professor M. CIPOLLA: Fourier's series; Dirichlet's problem; spherical and cylindrical functions; functions of a complex variable; elliptic functions; applications, four hours.—By Professor E. DANIELE: Differential equations of mathematical physics with applications, four hours.—By Professor G. SCORZA: Geometry of hyperspaces and some of its applications, four hours.—By Professor C. SEVERINI: Calculus of variations, four and one half hours.

UNIVERSITY OF GENOA.—By Professor G. LORIA: Infinitesimal geometry of curves and surfaces, three hours.—By Professor O. TEDONE: Partial differential equations with two independent variables and their application to the resolution of physical problems, three hours. By Professor ————: Analysis (advanced course), three hours.

UNIVERSITY OF MESSINA.—By Professor M. BOTTASSO: Vector analysis; newtonian potential; theory of elasticity, three hours.—By Professor P. CALAPSO: Elliptic functions, three hours.—By Professor G. GIAMBELLI: Analytic geometry of hyperspaces; geometrical theory of algebraic elimination, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics (from early times to 1200 A.D.), three hours.—By Professor A. DEL RE: n-dimensional analysis of Grassmann with applications to differential geometry and mechanics, four and one half hours.—By Professor R. MARCOLONGO: Fourier's series; spherical, cylindrical and Lamé's functions; applications, three hours.—By Professor D. MON-

TESANO: Theory of birational transformations of space; involutory birational transformations, three hours.—By Professor E. PASCAL: Selected topics of mathematical analysis, three hours.—By Professor L. PINTO: Geometrical optics, three hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Functions of a complex variable; integral equations, four hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Electromagnetic field, four hours.—By Professor G. RICCI: Absolute differential calculus with applications, four hours.—By Professor F. SEVERI: Differential geometry, four hours.

UNIVERSITY OF PALERMO: By Professor G. BAGNERA: General analytic functions; entire functions; linear differential equations, three hours.—By Professor M. DE FRANCHIS: Geometry of algebraic surfaces, three hours.—By Professor M. GEBBIA: Mechanics of continuous systems; newtonian and logarithmic potential; hydrostatics and hydrodynamics, four and one half hours.—By Professor A. SIGNORINI: Selected chapters of rational mechanics with particular regard to the theory of elasticity, three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Algebraic curves and surfaces, three hours.—By Professor U. CISOTTI: Hydrodynamics, three hours.—By Professor F. GERBALDI: Functions of a complex variable and elliptic functions, three hours.—By Professor G. VIVANTI: Theory of functions of real variables, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Geometry on an algebraic curve, three hours.—By Professor L. BIANCHI: Theory of the continuous groups of transformations, four and one half hours.—By Professor U. DINI: Fourier's series and more general developments concerning the analytic representation of functions of a real variable in a given interval, four and one half hours.—By Professor G. A. MAGGI: Steady and variable electromagnetic fields, four and one half hours.—By Professor ————: Theoretical mechanics (advanced course), three hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of calculus, three hours.—By Professor

E. BOMPIANI: Contact transformations of space and their continuous groups, three hours.—By Professor G. CASTELNUOVO: Algebraic equations and groups of substitutions, three hours.—By Professor U. CRUDELI: Theory of continuous groups of transformations, three hours.—By Professor L. SILLA: Differential equations of dynamics, three hours.—By Professor V. VOLTERRA: General wave theory, three hours; Hydrodynamics, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Equilibrium forms of a rotating fluid mass, three hours.—By Professor G. FUBINI: Modular, automorphic, fuchsian functions; linear differential equations with rational coefficients, three hours.—By Professor C. SEGRE: Algebraic complexes of straight lines, three hours.—By Professor C. SOMIGLIANA: Thermodynamics; theory of gases; propagation of heat, three hours.

PROFESSORS G. COLONNETTI, of the University of Pisa, E. LAURA, of the University of Pavia, and R. MARCOLONGO, of the University of Naples, has been elected corresponding members of the Reale Istituto Lombardo.

PROFESSOR P. HEEGAARD, of the University of Copenhagen, has been made professor of mathematics at the University of Christiania.

DR. T. BONNESEN has been appointed professor of descriptive geometry at the Copenhagen polytechnic school.

PROFESSOR E. HECKE, of the University of Basel, has been made professor of mathematics at the University of Göttingen.

PROFESSOR H. WEYL, of the polytechnic school of Zurich, has been made professor of mathematics at the University of Breslau.

SIR JOSEPH LARMOR has been awarded the Poncelet prize for the mathematical sciences this year by the Paris Academy of Sciences.

PROFESSOR HORACE LAMB, of the University of Manchester, has been appointed Halley lecturer at Oxford University for the coming year.

AT Clark University Professor A. G. WEBSTER has organized his department of physics for 1918–1919 into a ballistic institute, in which it is proposed to carry on research upon a considerable variety of subjects experimental as well as theoretical.

THE United States Bureau of Education has recently issued a Union List of Mathematical Periodicals prepared by Professor DAVID EUGENE SMITH and Dr. CAROLINE EUSTIS SEELY. This list contains the leading mathematical periodicals needed by research students and to be found in a number of the larger libraries in various parts of the country. Copies may be secured by addressing the United States Commissioner of Education, Washington, D. C.

THE following university and college teachers of mathematics have recently joined the national military service:

Professor ARNOLD DRESDEN, of the University of Wisconsin, Secretary of the Chicago Section of the Society, has gone to France in the service of the Red Cross. Professor W. C. GRAUSTEIN, of Rice Institute, has joined the ordnance at the Aberdeen proving ground. Dr. T. R. HOLLCROFT, of Columbia University, has entered the field artillery officers' training camp at Camp Zachary Taylor, Ky. Professor E. V. HUNTINGTON, of Harvard University, president of the Mathematical Association of America, has been given leave of absence and with the rank of major in the national army is assigned to statistical work under the chief of staff with residence at Washington. Professor P. R. RIDER, of Washington University, has entered the coast artillery training camp at Fort Monroe. Professor J. E. ROWE, of Pennsylvania State College, is engaged in mathematical research for the national advisory committee for aeronautics at Washington. Dr. NORBERT WIENER has joined the ordnance at the Aberdeen proving ground.

PROFESSOR G. A. MILLER, of the University of Illinois, has accepted the chairmanship of a committee which is to make a survey of the mathematical instruction given under the auspices of the Y. M. C. A. at the various naval stations.

PROFESSOR E. D. ROE, JR., of Syracuse University, has been given leave of absence for a year to devote his entire time to research.

AT the University of Illinois, the following promotions are announced: associate professor J. B. SHAW to a full professorship, assistant professor R. D. CARMICHAEL to an associate professorship, associates A. J. KEMPNER and G. E. WAHLIN to assistant professorships, instructor E. W. CHITTENDEN to associate, assistant R. F. BORDEN to an instructorship. Associate professor A. B. COBLE, of Johns Hopkins University, has been appointed professor of mathematics. Associate professor HENRY BLUMBERG, of the University of Nebraska, has been appointed associate and Mr. J. B. ROSENBACH, of the University of New Mexico, assistant. Assistant professor C. H. SISAM and Professor H. L. RIETZ have both resigned, Professor SISAM to become head of the department of mathematics at Colorado College and Professor RIETZ to become head of the department at the State University of Iowa.

DR. M. G. GABA, of Cornell University, has been appointed associate professor of mathematics at the University of Nebraska.

ASSISTANT professor W. V. LOVITT, of Purdue University, has been appointed associate professor of mathematics at Colorado College.

DR. FLORA E. LE STOURGEON, of the Liggett School, Detroit, has accepted an instructorship in mathematics at Mount Holyoke College.

DR. I. A. BARNETT has been appointed instructor in mathematics at Washington University.

ASSISTANT professor E. S. SMITH, of the University of Cincinnati, has been appointed acting commandant in addition to his duties in the department of mathematics.

PROFESSOR F. S. NOWLAN, of Brandon College, Brandon, Manitoba, has been appointed instructor in mathematics at Bowdoin College.

MR. J. W. LASLEY, JR., of the University of North Carolina, has been promoted to an assistant professorship of mathematics and given leave of absence for a year to study at the University of Chicago.

PROFESSOR G. CANTOR, of the University of Halle, died January 6, 1918, at the age of 73 years.

PROFESSOR M. SIMON, of the University of Strassburg, died in January, 1918, in his seventy-fourth year.

THE death is announced of Professor E. R. NEOVIUS, of the University of Helsingfors, at the age of 67 years.

THE death is announced of ALBERT GAUTHIER-VILLARS, head of the great publishing house of Paris.

PROFESSOR MAXIME BÔCHER, of Harvard University, died September 12, 1918, at the age of 51 years. Professor Bôcher had been a member of the American Mathematical Society since 1892, and was its president in 1909–1910.

THE Rev. Dr. G. M. SEARLE, superior general of the Paulist Fathers from 1904 to 1909, and previously professor of mathematics and director of the astronomical observatory of the Catholic University at Washington, died on July 8, 1918, at the age of 79 years. Dr. SEARLE had been a member of the American Mathematical Society since 1889.

PROFESSOR A. L. DANIELS, of the University of Vermont, died July 18, 1918, at the age of 69 years. Professor Daniels was made professor emeritus in 1914 after 29 years of service.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

EVANS, G. C. The Cambridge Colloquium, 1916, Part I; Functionals and their applications to selected topics, including integral equations. New York, American Mathematical Society, 1918. 8vo. 12 + 136 pp. Paper. $1.50; to members of the Society, $1.00

FORSYTH (A.R.). Functions of a complex variable. 3d edition. Cambridge, University Press, 1918. 24 + 855 pp. 30s.

LANDAU (E.). Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. Leipzig, Teubner, 1918. 8vo. 143 pp.

MANFREDI (L.). Geometría proyectiva y descriptiva según los programas de la Facultad de ciencias exactas, fisicas y naturales de la Universidad de Buenos Aires. Parte tercera: Applicaciones. Tercera edición, corregida y notablemente aumentada. (Texto y atlas.) Buenos Aires, Libreria cientifica y casa editora de Augusto Galli, 1917.

OPPERMANN (A.). Note sur le quadrilatère complet. Paris, Gauthier-Villars, 1918. 8vo. 27 pp.

ROGERS (A. L.). Experimental tests of mathematical ability and their prognostic value. (Teachers College, Columbia University, Contributions to Education, No. 89.) New York, 1918. 4 + 118 pp.

SCHOUTEN (G.). De beginselen van de leer der congruentiën. Groningen, P. Noordhoff, 1917. 8vo. 62 pp.

SEELY (C. E.). See SMITH (D. E.).

SETHE (K.). Von Zahlen und Zahlworten bei den alten Aegyptern; ein Beitrag zur Geschichte von Rechenkunst und Sprache. (Schriften der wissenschaftlichen Gesellschaft zu Strassburg, 25tes Heft.) Strassburg, Trübner, 1916.

SMITH (D. E.) and SEELY (C. E.). Union list of mathematical periodicals. (Bureau of Education, Bulletin, 1918, No. 9.) Washington, Government Printing Office, 1918. 8vo. 60 pp.

WIJDENES (P.). Uitgewerkte mondelinge examens hoogere algebra. Groningen, P. Noordhoff, 1917. 8vo. 51 pp. Fl. 5.00

ZEUTHEN (H. G.). Sur la réforme qu'a subie la mathématique de Platon à Euclide, et grâce à laquelle elle est devenue science raisonnée. (Extrait des *Mémoires de l'Académie royale des Sciences et des Lettres de Danemark:* Section des Sciences, 8e série, Tome 1, No. 5.) Copenhagen, Bianco Luno, 1917.

II. ELEMENTARY MATHEMATICS.

BROWN (J. C.). See WENTWORTH (G.).

ELMORE (E. B.). Regents original exercises in plane geometry selected in accordance with the list of propositions adopted by the New York State examinations board. Syracuse, N. Y., C. W. Bardeen, 1918. 8vo. $0.25

GONGGRIJP (B.). Theorie der rekenkunde, leerboek voor hoogere burger schoolen, gymnasia, kweek- en normaalscholen. Eerste stukje. Groningen, P. Noordhoff, 1917. 8vo. 69 pp. Gecart. Fl. 0.65

GONGGRIJP (B.) en WIJDENES (P.). Antwoorden op de vraagstukken van het leerboek der goniometrie en trigonometrie. Groningen, P. Noordhoff, 1917. 8vo. 38 pp. Fl. 0.60

——— Beknopte driehoeksmeting. Groningen, P. Noordhoff, 1917. 8vo. 138 pp. Gecart. Fl. 1.50

——— Antwoorden op de vraagstukken van de beknopte driehoeksmeting. Groningen, P. Noordhoff, 1917. 8vo. 29 pp. Fl. 0.50

HAWKES (H. E.), LUBY (W. A.) and TOUTON (F. C.). Second course in algebra. Revised edition. Boston, Ginn, 1918. 8 + 277 pp. Cloth. $1.00

LUBY (W. A.). See HAWKES (H. E.).

SCHLAUCH (W. S.). See WENTWORTH (G.).

SMITH (D. E.). See WENTWORTH (G.).

THORNDIKE (E. L.). The Thorndike arithmetics. Books 1–3. New York, Rand and McNally, 1917. 8vo. 16 + 262 + 16 + 288 + 16 + 330 pp. $0.48 + 0.54 + 0.60

TOUTON (F. C). See HAWKES (H. E.).

WENTWORTH (G.), SMITH (D. E.) and BROWN (J. C.). Junior high school mathematics. Book 3. Boston, Ginn, 1918. 6 + 282 pp. $0.96

WENTWORTH (G.), SMITH (D. E.) and SCHLAUCH (W. S.). Commercial algebra. Book 2. Boston, Ginn, 1918. 6 + 250 pp. Cloth. $1.12

WIJDENES (P.). See GONGGRIJP (B.).

III. APPLIED MATHEMATICS.

ASTROGRAPHIC catalogue. Hyderabad section. 1900.0. Vol. 1: Measures of rectangular coordinates and diameters of 63,436 star-images on plates with centres in dec. −170. Edinburgh, Neill, 1918. 43 + 223 pp. 16s. (12 rupees)

BOILEAU (C.). Le moteur à essence adapté à l'automobile et à l'aviation. Paris, Dunod et Pinat, 1918. 4to. 10 + 179 pp. Fr. 18.00

BRAHE (T.). Tychonis Brahe Dani opera omnia edidit I. L. E. Dreyer. Tomi quarti fasc. prior. Hauniæ, Libraria Gyldendaliana, 1918. 376 pp.

BUTLER (G. M.). A manual of geometrical crystallography. New York, Wiley, 1918. 8 + 155 pp. Cloth. $1.50

DREYER (I. L. E.). See BRAHE (T.).

DUMAS (A.). Les aéroplanes au salon de 1913. Paris, Gauthier-Villars, 1914. 8vo. 4 + 82 pp. Fr. 2.75

DUNCAN (J.) and STARLING (S. D.). A text-book of physics for the use of students of science and engineering. London, Macmillan, 1918. 23 + 1081 pp. 15s.

ECCLES (J. R.). Lecture notes on light. Cambridge, University Press, 1917. 215 pp. 12s. 6d.

FRENCH (J.W.). See STEINHEIL (A.).

GONGGRIJP (B.). Overzigt der mechanica, leerboek voor hoogere burgerscholen met 5-jarigen cursus. Groningen, P. Noordhoff, 1917. 8vo. 148 pp. Fl. 2.00

JAMES (W. H.) and MACKENZIE (M. C.). Principles of mechanism. New York, Wiley, 1918. 244 pp. Cloth. $1.50

MACKENZIE (M. C.). See JAMES (W. H.).

MIALARET (J. H. A.). Handleiding tot de practijk der scheefhoekige en der rechthoekige axonometrie. Amsterdam, L. J. Veen, 1917. 8vo. 136 pp.

MORLEY (A.). The theory of structures. London, Longmans, 1918. 9 + 584 pp. 14s.

NACHMAN (H. L.). Elements of machine design. New York, Wiley, 1918. 5 + 245 pp. Cloth. $2.00

PEELE (R.). Mining engineers' handbook. New York, Wiley, 1918. 2400 pp. $5.00

STARLING (S. D.). See DUNCAN (J.).

STEINHEIL (A.) and VOIT (E.). Applied optics: the computation of optical systems. Being the "Handbuch der angewandte Optik" of A. Steinheil and E. Voit translated and edited by J. W. French. Vol. 1. London, Blackie, 1918. 17 + 170 pp. 12s. 6d.

VIDALIE (R.). Etude générale du moteur rotatif. Paris, Dunod et Pinat, 1918. 8vo. 4 + 149 pp. Fr. 7.20

VILLAMIL (R. DE). Resistance of air. London, Spon, 1918. 10 + 192 pp. 7s. 6d.

VOIT (E.). See STEINHEIL (A.).

WIJDENES (P.). Beknopte beschrijven de meetkunde. Groningen, P. Noordhoff, 1917. 8vo. 94 pp. Gecart. Fl. 1.10